

THE MONGE-AMPÈRE EQUATION ON ALMOST COMPLEX MANIFOLDS

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ABSTRACT. We study the Dirichlet problem for the Monge-Ampère equation on almost complex manifolds. We obtain the existence of the unique smooth solution of this problem in strictly pseudoconvex domains.

Let (M, J) be an almost complex manifold of real dimension $2n$ (every definition will be given in section 1). N. Pali proved (in [P]) that, as it is in the case of complex geometry, for plurisubharmonic functions the $(1, 1)$ current $i\partial\bar{\partial}u$ is nonnegative. So for smooth plurisubharmonic functions u we have well defined Monge-Ampère operator $(i\partial\bar{\partial}u)^n \geq 0$ and we can study the complex Monge-Ampère equation $(i\partial\bar{\partial}u)^n = fdV$ where $f \geq 0$ and dV is a volume form (see (1.1) for this in local coordinates).

Let $\Omega \Subset M$ be a domain, ρ is a strictly plurisubharmonic function of class \mathcal{C}^2 in a neighborhood of $\bar{\Omega}$ (strictly plurisubharmonic means here that $(i\partial\bar{\partial}\rho)^n > 0$), such that $\Omega = \{\rho < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$, so we have a metric $\omega = i\partial\bar{\partial}\rho$ on Ω . In this article we study the following Dirichlet problem for the Monge-Ampère equation:

$$(1) \quad \begin{cases} u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\bar{\Omega}) \\ (i\partial\bar{\partial}u)^n = f\omega^n \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

where $f > 0$, $f, \varphi \in \mathcal{C}^\infty(\Omega)$. The main theorem is the following:

Theorem 1. *There is a unique smooth plurisubharmonic solution u of the problem (1).*

In [C-K-N-S] the above theorem was proved for J integrable.

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1. NOTION

We say that (M, J) is an almost complex manifold if M is a manifold and J is an (C^∞) smooth endomorphism of the tangent bundle TM , such that $J^2 = -\text{id}$. The real dimension of M is even in that case.

We have then a direct sum decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus_{(\mathbb{C})} T^{0,1}M$ where $T_{\mathbb{C}}M$ is a complexification of TM , $T^{1,0}M = \{X - iJX : X \in TM\}$ and $T^{0,1}M = \{X + iJX : X \in TM\} (= \{\zeta \in T_{\mathbb{C}}M : \bar{\zeta} \in T^{0,1}M\})$.

Let \mathcal{A}^k be the set of k -forms i.e. the set of sections of $\bigwedge^k(T_{\mathbb{C}}M)^*$ and $\mathcal{A}^{p,q}$ be the set of (p, q) -forms i.e. the set of sections of $\bigwedge^p(T^{1,0}M)^* \otimes_{(\mathbb{C})} \bigwedge^q(T^{0,1}M)^*$. Then we have a direct sum decomposition $\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$. We denote the projections $\mathcal{A}^k \rightarrow \mathcal{A}^{p,q}$ by $\Pi^{p,q}$.

If $d : \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$ is (the \mathbb{C} -linear extension of) the exterior differential, then we define $\partial : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p+1,q}$ as $\Pi^{p+1,q} \circ d$ and $\bar{\partial} : \mathcal{A}^{p,q} \rightarrow \mathcal{A}^{p,q+1}$ as $\Pi^{p,q+1} \circ d$.

We say that an almost complex structure J is integrable if satisfy any of following (equivalent) conditions:

- i) $d = \partial + \bar{\partial}$.
- ii) $\bar{\partial}^2 = 0$.
- iii) $[\zeta, \xi] \in T^{0,1}$ for vector fields $\zeta, \xi \in T^{0,1}$.

By the Newlander-Nirenberg Theorem J is integrable if and only if it is induced by a complex structure.

In the paper ζ_1, \dots, ζ_n is always a (local) frame of $T^{1,0}$. Let us put for a smooth function u

$$u_{p\bar{q}} = \zeta_p \bar{\zeta}_q u = u_{\bar{q}p} + [\zeta_p, \bar{\zeta}_q]u$$

and

$$A_{p\bar{q}} = A_{p\bar{q}}(u) = u_{p\bar{q}} - [\zeta_p, \bar{\zeta}_q]^{0,1}u$$

where $X^{0,1} \in T^{0,1}$ is such that $X - X^{0,1} \in T^{1,0}$ for any $X \in T_{\mathbb{C}}M$. Then for a smooth function u we have (see [P]):

$$i\partial\bar{\partial}u = i \sum A_{p\bar{q}} d\zeta_p \wedge d\bar{\zeta}_q.$$

So locally we can write:

$$(1.1) \quad \text{MA}u = \det(A_{p\bar{q}}) = \tilde{f}$$

where $\tilde{f} = f$ if ζ_1, \dots, ζ_n are orthonormal.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We say that a (smooth) function $\lambda : \mathbb{D} \rightarrow M$ is J -holomorphic or simpler holomorphic if $\lambda'(d\bar{z}) \in T^{0,1}T$. The plenty of such disks show the following proposition from [I-R], where is stated for $C^{k,\alpha}$ class of J :

Proposition 1.1. *Let $0 \in M \subset \mathbb{R}^{2n}$, $k, k' \geq 1$. For $v_0, v_1, \dots, v_k \in \mathbb{R}^{2n}$ close enough to 0 there is holomorphic function $\lambda : \mathbb{D} \rightarrow M$ such that $\lambda(0) = v_0$ and $\frac{\partial^l \lambda}{\partial x^l} = v_l$, for $l = 1, \dots, k$. Moreover we can choose λ with \mathcal{C}^1 dependence on parameters $(v_0, \dots, v_k) \in (\mathbb{R}^{2n})^{k+1}$, where for holomorphic functions we consider $\mathcal{C}^{k'}$ norm.*

An upper semi-continuous function u on an open subset of M is said to be plurisubharmonic if a function $u \circ \lambda$ is subharmonic for every holomorphic function λ . We denote set of plurisubharmonic functions on $\Omega \subset M$ by $\mathcal{PSH}(\Omega)$. For smooth functions u it means that matrix $(A_{p\bar{q}})$ is nonnegative.

2. COMPARISON PRINCIPLE

In this section $\Omega \Subset M$ is a domain not necessary strictly pseudoconvex.

Proposition 2.1. *If $u, v, \rho \in \mathcal{C}^2(\bar{\Omega})$ are plurisubharmonic functions, with $i\partial\bar{\partial}\rho > 0$ in Ω and such that $\text{MA}u \geq \text{MA}v$ and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.*

Proof: First, let us assume that $\text{MA}u > \text{MA}v$. As in [C-K-N-S], in a point where $\nabla(u - v) = 0$, we have

$$0 < \text{MA}u - \text{MA}v = \int_0^1 \frac{d}{dt} \text{MA}(tu - (1-t)v) dt = \left(\int_0^1 B^{p\bar{q}}(t) dt \right) (u - v)_{p\bar{q}},$$

where $(B^{p\bar{q}}(t))$ is the transpose of the inverse of the matrix $((tu - (1-t)v)_{p\bar{q}} - [\zeta_p, \bar{\zeta}_q]^{0,1}(tu - (1-t)v))$. Thus the function $u - v$ attains his maximum on the boundary of Ω .

In the general case we put $u' = u + \varepsilon(\rho - \sup_{\bar{\Omega}} \rho)$ and the lemma follows from the above case (with u' instead of u). \square

3. A PRIORI ESTIMATE

In this section we will proof a $\mathcal{C}^{1,1}$ estimate for the smooth solution u of problem (1). By the general theory of elliptic equations (see for example [C-K-N-S]) we obtain from this the $\mathcal{C}^{k,\alpha}$ estimate and then the existence of smooth solution. The uniqueness follows from comparison principle.

Our proofs are close to [C-K-N-S] but more complicated because of noncommutativity of some vector fields.

3.1. some technical preparation. In this section we assume that $\Omega \Subset M$ is strictly pseudoconvex i.e. on the neighborhood of $\bar{\Omega}$ there is a plurisubharmonic function ρ such that $\Omega = \{\rho < 0\}$ and $\nabla \rho \neq 0$ on $\partial\Omega$.

We have for X, Y vektor fields

$$(3.1) \quad X(\log f) = A^{p\bar{q}} X A_{p\bar{q}},$$

$$(3.2) \quad XY(\log f) = A^{p\bar{q}} XY A_{p\bar{q}} - A^{p\bar{j}} A^{i\bar{q}} (Y A_{i\bar{j}}) (X A_{p\bar{q}}).$$

where $(A^{p\bar{q}})$ is the inverse of the matrix $(\overline{A_{p\bar{q}}})$.

Let us define elliptic operator $L = L_\zeta = A^{p\bar{q}}(\zeta_p \bar{\zeta}_q - [\zeta_p, \bar{\zeta}_q])$.

We will often use the following fact:

$$|Xu_k|^2, |Xu_{\bar{k}}|^2 \leq C(|\nabla u|^2 + \sum_p |u_{pk}|^2 + |u_{\bar{p}k}|^2)$$

for the smooth vector fields X where $C = C(\|X\|, \|[X, \zeta_k]\|, \|[X, \bar{\zeta}_k]\|)$.

We will change coordinates a lot but we will always have this under control.

We will use the fact that for every $A > 0$ we can choose $\psi > 0$ such that $\psi_{p\bar{p}} - [\zeta_p, \bar{\zeta}_p]^{0,1} \geq A(|\psi_p|^2 + \psi)$.

In the proofs below C is a constant under control, but it can change from a line to a next line.

3.2. uniform estimate.

Lemma 3.1. *We have $\|u\|_{L^\infty(\Omega)} \leq C$, where $C = C(\|\rho\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)})$.*

Proof: From the comparison principle and the maximum principle we have

$$\|f\|_{L^\infty(\Omega)} \rho + \inf_{\partial\Omega} \varphi \leq u \leq \sup_{\partial\Omega} \varphi. \quad \square$$

3.3. gradient estimate. In two next lemmas we shall prove *a priori* estimate for first derivate.

Lemma 3.2. *We have*

$$\|u\|_{C^{0,1}(\partial\Omega)} \leq C,$$

where $C = C(\omega, \|f\|_{L^\infty(\Omega)}, \|\varphi\|_{C^{1,1}(\Omega)})$.

Proof: We can choose $A > 0$ such that $A\omega + i\partial\bar{\partial}\varphi \geq f\omega$ and $A\omega \geq i\partial\bar{\partial}\varphi$. Thus by the comparison principle and the maximum principle we have

$$\varphi + A\rho \leq u \leq \varphi - A\rho$$

for A large enough. So on the boundary we have

$$|\nabla u| \leq |\nabla A\rho| + |\nabla \varphi|. \quad \square$$

Lemma 3.3. *We have*

$$(3.3) \quad \|u\|_{C^{0,1}(\Omega)} \leq C,$$

where $C = C(\Omega, \|f^{1/n}\|_{C^{0,1}}, \|f\|_{L^\infty(\Omega)})$.

Proof: Consider the function $v = \psi|\nabla u|^2$. We assume that v takes its maximum in $z_0 \in \Omega$. We can choose ζ_1, \dots, ζ_n , such that they are orthonormal in a neighbourhood of z_0 , and the matrix $A_{p\bar{q}}$ is diagonal at z_0 . From now on all formulas are assumed to hold at z_0 .

We have $Xv = 0$ so that $X(|\nabla|^2) = -X \log \psi |\nabla u|^2$. We can calculate

$$\begin{aligned} L(v) &= L(\psi)|\nabla|^2 + \psi L(|\nabla|^2) + A^{p\bar{p}}(\psi_p(|\nabla|^2)_{\bar{p}} + \psi_{\bar{p}}(|\nabla|^2)_p) \\ L(|\nabla|^2) &= A^{p\bar{p}}(|\nabla|^2)_{p\bar{p}} - [\zeta_p, \bar{\zeta}_p]^{0,1} |\nabla|^2 \\ &= A^{p\bar{p}} \sum_k (u_{p\bar{p}k} u_{\bar{k}} + u_k u_{p\bar{p}\bar{k}} + |u_{pk}|^2 + |u_{\bar{p}k}|^2 - [\zeta_p, \bar{\zeta}_p]^{0,1} u_k u_{\bar{k}} - u_k [\zeta_p, \bar{\zeta}_p]^{0,1} u_{\bar{k}}) \\ &\quad - A^{p\bar{p}} (u_{p\bar{p}k} - [\zeta_p, \bar{\zeta}_p]^{0,1} u_k) \\ &= A^{p\bar{p}} (u_{kp\bar{p}} - \zeta_k [\zeta_p, \bar{\zeta}_p]^{0,1} u + \zeta_p [\bar{\zeta}_p, \zeta_k] u + [\zeta_p, \zeta_k] \bar{\zeta}_p u) \\ &= (\log f)_k + A^{p\bar{p}} (\zeta_p [\bar{\zeta}_p, \zeta_k] u + \bar{\zeta}_p [\zeta_p, \zeta_k] u + [[\zeta_p, \zeta_k], \bar{\zeta}_p] u - [[\zeta_p, \bar{\zeta}_p]^{0,1}, \zeta_k] u) \end{aligned}$$

then we have

$$|A^{p\bar{p}}(u_{p\bar{p}k} - [\zeta_p, \bar{\zeta}_p]^{0,1} u_k)| < C \left(\frac{\|f^{1/k}\|_{C^{0,1}}}{f^{1/n}} + A^{p\bar{p}} \left(\sum_s (|u_{ps}| + |u_{p\bar{s}}|) + |\nabla u| \right) \right)$$

and similarly

$$|A^{p\bar{p}}(u_{p\bar{p}\bar{k}} - [\zeta_p, \bar{\zeta}_p]^{0,1} u_{\bar{k}})| < C \left(\frac{\|f^{1/k}\|_{C^{0,1}}}{f^{1/n}} + A^{p\bar{p}} \left(\sum_s (|u_{ps}| + |u_{p\bar{s}}|) + |\nabla u| \right) \right)$$

so for the proper choice of ψ we have $L(v)(0) > 0$ and this is a contradiction with the maximality of v . \square

3.4. $\mathcal{C}^{1,1}$ estimate. Let $P \in \partial\Omega$. Estimate of $XYu(P)$ where X, Y are tangent to $\partial\Omega$ follows from the gradient estimate.

Lemma 3.4. *We have*

$$(3.4) \quad \|NTu(P)\| \leq C,$$

where a vector field T is tangent to $\partial\Omega$ and N is a vector normal to $\partial\Omega$. $C = C(\Omega, \|f^{1/n}\|_{C^{0,1}}, \|f\|_{L^\infty(\Omega)}, \|N\|, \|T\|_{C^{0,1}})$.

Proof: X_1, X_2, \dots, X_{2n} be orthonormal vector fields near P such that $X_{2k} = JX_{2k-1}$ and X_{2n} is normal to the boundary near P . Let X be a vector field tangent to the boundary. Consider the function $v = X(u - \varphi) + B\rho + \sum_{k=1}^{2n-1} |X_k(u - \varphi)|^2 - A(d(\cdot, \cdot))^2$. For A large enough $v \leq 0$ on the boundary of Ω .

Our goal is to show that for B large enough we have $L(Xu) \geq 0$. Let us calculate:

$$\clubsuit \quad L(X(u - \varphi)) \geq L(Xu) - C \sum A^{p\bar{p}},$$

$$\begin{aligned} L(Xu) &= A^{p\bar{q}}(\zeta_p \bar{\zeta}_q Xu - [\zeta_p, \bar{\zeta}_q]^{0,1} Xu) \\ &= X \log f + A^{p\bar{q}}(\zeta_p [\bar{\zeta}_q, X]u + [\zeta_p, X] \bar{\zeta}_q u - [[\zeta_p, \bar{\zeta}_q]^{0,1}, X]u) \\ [\bar{\zeta}_q, X] &= \alpha_q \bar{\zeta}_n + \sum_{s=1}^{2n-1} \alpha_{q,s} X_s \end{aligned}$$

and so

$$A^{p\bar{q}}(\zeta_p [\bar{\zeta}_q, X]u) = \sum_q \alpha_q \delta_{qn} + \sum_{s=1}^{2n-1} \alpha_{q,s} A^{p\bar{q}} \zeta_p X_s u + Yu$$

where Y is a vector field which gives

$$|A^{p\bar{q}} \zeta_p [\bar{\zeta}_q, X]u| \leq C \sum A^{p\bar{p}} (1 + \sum_{s=1}^{2n-1} |\alpha_{q,s} \zeta_p X_s u|)$$

In a similar way we can calculate

$$A^{p\bar{q}}[\zeta_p, X] \bar{\zeta}_q u$$

and we obtain

$$L(X(u - \varphi)) \geq -C \sum_{p,s} A^{p\bar{p}} (1 + |\zeta_p X_s u|).$$

$$\diamond \quad L\left(\sum_{k=1}^{2n-1} |X_k(u - \varphi)|^2\right)$$

$$= A^{p\bar{q}} \sum_{k=1}^{2n-1} ((\zeta_p X_k(u - \varphi))(\bar{\zeta}_q X_k(u - \varphi)) + (\zeta_p \bar{\zeta}_q X_k(u - \varphi))X_k(u - \varphi))$$

Similarly as in \clubsuit we have

$$A^{p\bar{q}} \zeta_p \bar{\zeta}_q X_k u = X_k \log f + A^{p\bar{q}}(\zeta_p [\bar{\zeta}_q, X_k]u + [\zeta_p, X_k] \bar{\zeta}_q u - [[\zeta_p, \bar{\zeta}_q]^{0,1}, X_k]u)$$

and

$$L\left(\sum_{k=1}^{2n-1} |X_k(u - \varphi)|^2\right)$$

$$\geq \frac{1}{2} A^{p\bar{q}} \sum_{k=1}^{2n-1} (\zeta_p X_k u)(\bar{\zeta}_q X_k u) - C A^{p\bar{p}} (1 + \sum_{k=1}^{2n-1} |\zeta_p X_k u|).$$

$$\heartsuit \quad L(\rho) \geq C^{-1} \sum A^{p\bar{p}}.$$

$$\spadesuit \quad L((d(\cdot, \cdot))^2) \geq -C \sum A^{p\bar{p}}.$$

Now we can conclude that for B large enough, since $L(X_1 u) \geq 0$, we have $v \leq 0$ on Ω and so $X_{2n} X u(P) \leq C \quad \square$.

Lemma 3.5. *We have*

$$(3.5) \quad \|N N u(P)\| \leq C,$$

where N is a vector field normal to $\partial\Omega$ $C = C(\Omega, \|f^{1/n}\|_{C^{1,1}}, \|N\|_{C^{0,1}})$.

Proof: By the previous Lemma it is enough to prove that $|\zeta|^2 \leq C \zeta \bar{\zeta} u(P) - [\zeta, \zeta]^{0,1}(P)$ for every vector field ζ orthogonal (at P) to N . We can assume $\rho_N(P) = 1$ and because our argue will be local we can assume also that $P = 0 \in \mathbb{C}^n$. Let $\zeta_1, \zeta_2, \dots, \zeta_n \in T^{1,0}$ be a orthonormal frame in a neighborhood of 0 such that $\zeta_n \rho = -1$. We can assume that $\zeta_1 = \zeta$. From the strictly pseudoconvexity and using the proposition 1.1 (for $k = 2$) we can choose J -holomorphic disk λ such that $\lambda(0) = 0$, $\frac{\partial \lambda}{\partial z}(0) = a \zeta_1$ for some $a > 0$ and

$$(3.6) \quad \text{dist}(\lambda(z), \bar{\Omega}) = b|z|^2 + O(|z|^3) \text{ for some } b > 0.$$

Now changing coordinates we may assume $\lambda(z_1) = (2z_1, 0)$, $\zeta_k(0) = \frac{\partial}{\partial z_k}$ for $k = 1, \dots, n$.

We can find a holomorphic cubic polynomial p such that

$$\begin{aligned} \varphi(z) &= \varphi(0) + \varphi'(0)(z) \\ &+ \text{Re} \sum_{p=1}^n \frac{\partial^2 \varphi}{\partial z_1 \partial \bar{z}_p} z_1 \bar{z}_p + \text{Re} p(z) + \alpha z_1 |z_1|^2 + O(|z_1|^4 + |z_2|^2 + \dots + |z_n|^2). \end{aligned}$$

By (3.6) we have $\text{Re} z_n = \beta |z_1|^2 + \beta' \text{Re} z_1^2 + O(|z_1|^3)$ on $\partial\Omega$ for some $\beta, \beta' > 0$ and we obtain

$$\begin{aligned} u - u(0) &= \varphi - \varphi(0) \\ &\leq \varphi'(0)(z) - \gamma \text{Re} z_n + \text{Re} \sum_{p=2}^n a_p z_1 \bar{z}_p + \text{Re} \tilde{p}(z) + O(|z_2|^2 + \dots + |z_n|^2) \end{aligned}$$

$$= \operatorname{Re} \sum_{p=2}^n a_p z_1 \bar{z}_p + \operatorname{Re} \tilde{p}(z) + O(|z_2|^2 + \dots + |z_n|^2)$$

on $\partial\Omega$ for some $\gamma \in \mathbb{R}$, $a_2, \dots, a_n \in \mathbb{C}$ and new cubic polynomials $\tilde{p}, \tilde{\bar{p}}$.

Let $B > 0$. By the proposition 1.1 there is a family of disks $g_w : \mathbb{D} \rightarrow \mathbb{C}^n$, $w \in \mathbb{C}^{n-1}$ such that $g_w(0) = (0, w)$, $\frac{\partial g_w}{\partial z}(0) = (1, \frac{a_2}{B}, \dots, \frac{a_n}{B})$, $\frac{\partial^2 g_w}{\partial x^2}(0) = 0$ and a function $G : \mathbb{C}^{n-1} \times \mathbb{D} \rightarrow \mathbb{C}^n$ given by $G(w, z) = g_w(z)$ is of class \mathcal{C}^2 .

Let $h(g_w(z)) = p_w(z) + AB|w|^2 + \varepsilon\rho$ where p_w is a holomorphic cubic polynomial in one variable such that $\operatorname{Re} \tilde{p}(h_w(z)) = \operatorname{Re} p_w(z) + \operatorname{Re} a_w z |z|^2 + O(|z|^4)$, $a_w \in \mathbb{C}$, $A, \varepsilon > 0$. If A is enough large then $h \geq u - u(0)$ on $\partial\Omega \cap U$ where U is a small neighborhood of 0. Enlarging A again we can assume $h \geq u - u(0)$ on ∂S where $S = \Omega \cap U$. Let $M > \|h\|_{\mathcal{C}^2(S)}$ for every $0 \leq \varepsilon \leq 1$. We can put $\varepsilon = \inf_S f M^{-n}$ and we get an inequality $(i\partial\bar{\partial}h)^n < (i\partial\bar{\partial}u)^n$ on set $S \cap \{i\partial\bar{\partial}h > 0\}$. This by the Comparison Principle (Proposition 2.1) gives us $h \geq u - u(0)$ on S . Now by $h_N > u_N$, $\gamma\rho_{1\bar{1}} = -\varphi_{1\bar{1}}$ and $u_{1\bar{1}} - \varphi_{1\bar{1}} = (\varphi_N - u_N)\rho_{1\bar{1}}$ we can conclude that $u_{1\bar{1}} \geq \varepsilon\rho_{1\bar{1}}$ \square

Lemma 3.6. *We have*

$$(3.7) \quad \|Hu\| \leq C,$$

where Hu is a Hessian of u , $C = C(\Omega, \|f^{1/n}\|_{\mathcal{C}^{1,1}})$.

Proof:

Let us define M as the biggest eigenvalue of the Hessian Hu . We will show that the function

$$\Lambda = \psi e^{K|\nabla u|^2} M,$$

where K^{-1} is large enough, doesn't attain maximum in Ω .

Assume that a maximum of the function Λ is attained at $z_0 \in \Omega$ (otherwise we are done). There are $\zeta_1, \dots, \zeta_n \in T_{z_0}^{1,0}$ orthonormal z_0 such that the matrix $(A_{p\bar{q}})$ is diagonal at z_0 . Let $X \in TM_{z_0}$ be such that $\|X\| = 1$ and $M = H(X, X)$. We can normalize coordinates near z_0 such that $z_0 = 0 \in \mathbb{C}^n$, $X = \frac{\partial}{\partial x_1}(0)$ and $J(z, 0) = J_{st}$ for small $z \in \mathbb{C}$. Then we can in a natural way extend ζ_1, \dots, ζ_n to some neighborhood U of 0 such that $[\zeta_k, X] = 0$ and $[\zeta_k, \zeta_k] = 0$ for $k = 1, \dots, n$ (first on $U \cap \mathbb{C} \times \{0\}$ as the same linear combination of vectors $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ as in 0, then we can take (pseudo) holomorphic disks $d_k : \mathbb{C} \supset \Delta \rightarrow U$ (for ζ_k not tangent $U \cap \mathbb{C} \times \{0\}$) such that $d_k(0) = 0$ and $\frac{\partial d_k}{\partial z}(0) = \zeta_k(0)$, and on image of d_k we can put $\zeta_k(w) = \frac{\partial d_k}{\partial z}(d_k^{-1}(w))$, on the end we extend the vector fields on whole U). Then the function $v = \psi e^{K|\nabla u|^2} \frac{Hu(\partial x_1, \partial x_1)}{|\partial x_1|^2} =$

$\Psi e^{K|\nabla u|^2}(u_{x_1 x_1} + Tu)$, where $\Psi = \frac{\psi}{|\partial x_1|^2}$ and T is a vector field, also have maximum at 0. Near 0 put $\mu = u_{x_1 x_1} + Tu$ (then $\mu(0) = M(0)$) and extend X as $X = \frac{\partial}{\partial x_1}$.

We will estimate $L(v)$:

$$\begin{aligned}
0 \leq L(v) &= L(\Psi e^{K|\nabla u|^2})\mu + \Psi e^{K|\nabla u|^2} L(\mu) - 2A^{p\bar{p}} \frac{(\Psi e^{K|\nabla u|^2})_p (\Psi e^{K|\nabla u|^2})_{\bar{p}}}{\Psi e^{K|\nabla u|^2}} \\
&\quad \blacklozenge L(\Psi e^{K|\nabla u|^2}) \\
&= K e^{K|\nabla u|^2} A^{p\bar{p}} (K^{-1} \Psi_{p\bar{p}} + \Psi_p (|\nabla u|^2)_{\bar{p}} + \Psi_{\bar{p}} (|\nabla u|^2)_p + \Psi (|\nabla u|^2)_{p\bar{p}} + K \Psi (|\nabla u|^2)_p)^2 \\
&\quad A^{p\bar{p}} (|\nabla u|^2)_{p\bar{p}} \\
&= A^{p\bar{p}} \sum_k ((\zeta_p \bar{\zeta}_p \eta_k u) u_{\bar{k}} + u_k (\zeta_p \bar{\zeta}_p \bar{\eta}_k u) + (\bar{\zeta}_p \eta_k u) (\zeta_p \bar{\eta}_k u) + (\zeta_p \eta_k u) (\bar{\zeta}_p \bar{\eta}_k u)) \\
&\quad = \sum_k ((\log f)_{\bar{k}} + (\log f)_{\bar{k}}) \\
&\quad + A^{p\bar{p}} \sum_k ((\zeta_p [\zeta_{\bar{p}}, \eta_k] u) u_{\bar{k}} + ([\zeta_p, \eta_k] u_{\bar{p}}) u_{\bar{k}} + (\zeta_p [\bar{\zeta}_p, \bar{\eta}_k] u) u_k + ([\zeta_p, \bar{\eta}_k] u_{\bar{p}}) u_k + (\eta_k + \bar{\eta}_k) [\zeta_p, \bar{\zeta}_p]^{0,1} u) \\
&\quad + A^{p\bar{p}} \sum_k ((\bar{\zeta}_p \eta_k u) (\zeta_p \bar{\eta}_k u) + (\zeta_p \eta_k u) (\bar{\zeta}_p \bar{\eta}_k u))
\end{aligned}$$

so we have

$$L(\Psi e^{K|\nabla u|^2}) \geq A^{p\bar{p}} \left(C + K \sum_k ((\bar{\zeta}_p \zeta_k u) (\zeta_p \bar{\zeta}_k u) + (\zeta_p \zeta_k u) (\bar{\zeta}_p \bar{\zeta}_k u)) \right)$$

$$\star L(\mu) = L(u_{x_1 x_1}) + L(Tu)$$

$$L(Tu) \leq T(\log f) - C(\mu + 1) \sum A^{p\bar{p}}$$

$$\begin{aligned}
L(u_{x_1 x_1}) &= (\log f)_{x_1 x_1} + A^{p\bar{p}} A^{q\bar{q}} |X(\zeta_p \bar{\zeta}_q - [\zeta_p, \bar{\zeta}_q]^{0,1}) u|^2 \\
&+ A^{p\bar{p}} (\zeta_p [\bar{\zeta}_p, X] Xu + [\zeta_p, X] \bar{\zeta}_p Xu + X \zeta_p [\bar{\zeta}_p, X] u + X [\zeta_p, X] \bar{\zeta}_p u + X X [\zeta_p, \bar{\zeta}_p]^{0,1} u) \\
&= (\log f)_{x_1 x_1} + A^{p\bar{p}} A^{q\bar{q}} |X(\zeta_p \bar{\zeta}_q - [\zeta_p, \bar{\zeta}_q]^{0,1}) u|^2 \\
&+ A^{p\bar{p}} ([\zeta_p, [\bar{\zeta}_p, X]] Xu + X [\zeta_p, [\bar{\zeta}_p, X]] u + [X, [\bar{\zeta}_p, X]] \zeta_p u + [X, [\zeta_p, X]] \bar{\zeta}_p u) \\
&\quad + A^{p\bar{p}} (X[X, [\zeta_p, \bar{\zeta}_p]^{0,1}] u + [X, [\zeta_p, \bar{\zeta}_p]^{0,1}] Xu) \\
&\geq -C(\mu + 1) \sum A^{p\bar{p}}
\end{aligned}$$

and

$$L(\mu) \geq -C(\mu + 1) \sum A^{p\bar{p}} ((\bar{\zeta}_p \zeta_k u) (\zeta_p \bar{\zeta}_k u) + (\zeta_p \zeta_k u) (\bar{\zeta}_p \bar{\zeta}_k u))$$

$$\blacksquare - 2A^{p\bar{p}} \frac{(\Psi e^{K|\nabla u|^2})_p (\Psi e^{K|\nabla u|^2})_{\bar{p}}}{\Psi e^{K|\nabla u|^2}}$$

$$\begin{aligned}
&= -2A^{p\bar{p}}e^{K|\nabla u|^2}\left(\frac{\Psi_p\Psi_{\bar{p}}}{\Psi}+K\Psi_p(|\nabla u|^2)_{\bar{p}}+K\Psi_{\bar{p}}(|\nabla u|^2)_p+K^2\Psi(|\nabla u|^2)_{\bar{p}}(|\nabla u|^2)_p\right) \\
&\geq -CA^{p\bar{p}}\left(C'+\sum_k(|(\zeta_p\zeta_k u)|+|(\bar{\zeta}_p\bar{\zeta}_k u)|+K^2((\bar{\zeta}_p\zeta_k u)(\zeta_p\bar{\zeta}_k u)+(\zeta_p\zeta_k u)(\bar{\zeta}_p\bar{\zeta}_k u)))\right)
\end{aligned}$$

We can conclude $L(v)(0) > 0$ and it is a contradiction with the maximality of v . \square

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